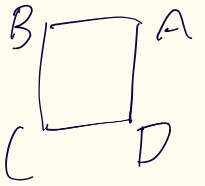
ISOMORPHISM THEOREMS

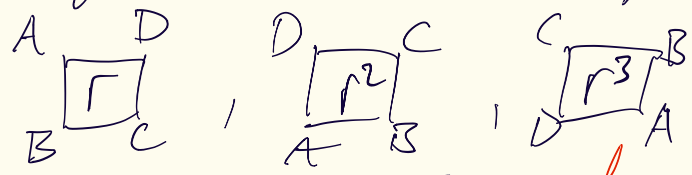
Isomorphism theorems

Recall $\varphi: G \to H$ is a hom. if $\varphi(a \cdot b) = \varphi(a)\varphi(b)$

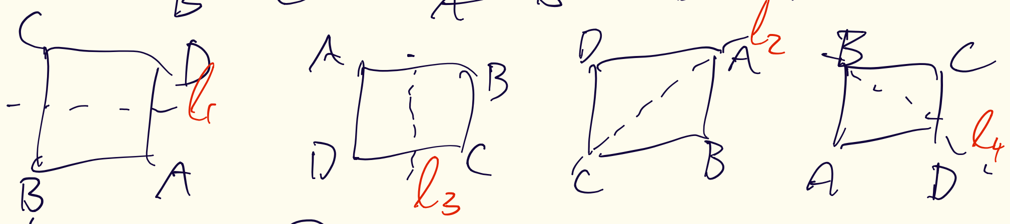
Consider the group of symmetries of a square



Rotations: e,



Flips:

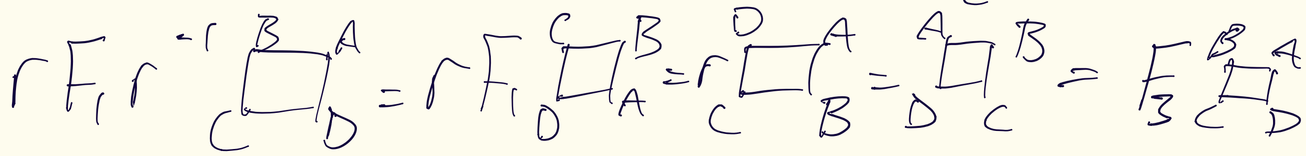


This is a Dihedral group D_8

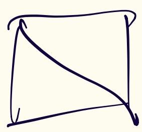
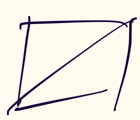
$C_4 = \langle r_{\frac{\pi}{2}} \rangle$ is a subgroup of D_8 (isomorphic to...)

Since $|D_8| / |C_4| = 2$, C_4 is normal in D_8

$C_2 = \langle F_1 \rangle < D_8$ $\quad r\langle F_1 \rangle r^{-1} = \{e, rF_1r^{-1}\}$



So $r\langle F_1 \rangle r^{-1} \neq \langle F_1 \rangle$ and so $\langle F_1 \rangle$ not normal in D_8

Consider now the elts of D_8 preserving the structure  or breaking it 

e, r^2, F_2, F_4

r, r^3, F_1, F_3

Consider a homomorphism $f: D_8 \to \langle F_1 \rangle$ defined as follows.

$$f: \begin{cases} e, r^2, F_2, F_4 \mapsto e \\ r, r^3, F_1, F_3 \mapsto F_1 \end{cases}$$

If two elts preserve the diagonal symmetry then their composition preserves it. $(e \cdot e = e)$

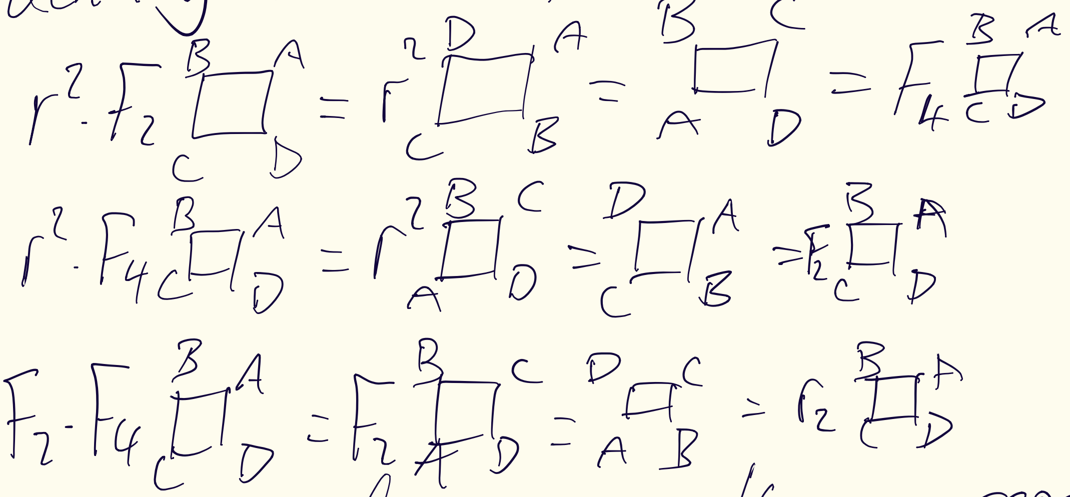
If elts that preserve & break the symmetry then their comp. breaks it $(e \cdot F_1 = F_1)$

If two elements break the symmetry then their composition preserves it $(F_1 \cdot F_1 = e)$.

So f is a homomorphism

Consider $\{e, r^2, F_2, F_4\} \subseteq D_4$. Is this a subgp?

identity is in ✓, every element is an involution so inverses ✓



So '$\circ$' is a binary operation, so assoc. is inherited.

Again this gp is normal in D_8 as its order is $\frac{8}{2}$

Call it $\ker f = \{e, r^2, F_2, F_4\}$

Note that f sends every elt of $\ker f$ to the identity in $\langle F_1 \rangle$

Def^n. Let $\varphi: G \to H$ be a hom. between groups. The kernel of φ is the subset of G

that gets mapped to the identity of H by φ:

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = e_H\}$$

$f: D_8 \to \mathbb{Z}_2$, f preserves the triangles.

$$\ker f = \{e, r^2, f_2, f_4\} \trianglelefteq D_8$$

Since, in this case, $\ker f \trianglelefteq D_8$ we can consider

$$D_8/\ker f = \left\{ \begin{matrix} \ker f \\ r \ker f \end{matrix} \right\} = \left\{ \begin{matrix} \{e, r^2, f_2, f_4\} \\ \{r, r^3, f_1, f_3\} \end{matrix} \right\} \cong \mathbb{Z}_2$$

Hence $D_8/\ker f \cong \mathbb{Z}_2$.

We will see that this is the general situation.

We are going to build a connection between homomorphisms & normal subgps.

Recall for $\varphi: G \to H$, $\varphi(e_G) = e_H$ & $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Let $N \lhd G$ & consider the map $\varphi: G \to G/N$ sending g to the coset gN.

Claim: φ is surjective. If $gN \in G/N$ then $g \in G$ so $\varphi(g) = gN$ ✓

Claim: φ hom. $\varphi(gh) = (gh)N = (gN) * (hN) = \varphi(g) * \varphi(h)$ ✓

Def'n $\varphi: G \to G/N : g \mapsto gN$ is called the <u>natural homomorphism</u> of G onto G/N

So every normal subgp has a corresponding hom.

Now we connect each hom to a normal subgp, using the kernel.

<u>Thm</u>: Let $\varphi: G \to H$ be a hom. $\ker(\varphi) \lhd G$

<u>Proof</u>: We show $\ker \varphi < G$ first.

For $g_1, g_2 \in G$ w/ $\varphi(g_1) = e_H = \varphi(g_2)$

$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) = e_H^2 = e_H$ So $g_1 g_2 \in \ker \varphi$

So '·' is a binary op.: $\ker\varphi \times \ker\varphi \to \ker\varphi$

Since $\varphi(e_G) = e_H$ we have $e_G \in \ker\varphi$

If $g \in \ker\varphi$ then $\varphi(e_G) = \varphi(gg^{-1}) = \varphi(g)\varphi(g^{-1}) = e_H$

$\varphi(g) = e_H$ So $\varphi(g^{-1}) = e_H$ So inverses in $\ker\varphi$.

if $g_1, g_2, g_3 \in \ker\varphi$ then

$$\varphi(g_1(g_2 g_3)) = \varphi(g_1)\varphi(g_2 g_3) = \varphi(g_1)(\varphi(g_2)\varphi(g_3)) = e_H \cdot e_H = \varphi(g_1 g_2)\varphi(g_3)$$
$$= \varphi((g_1 g_2) g_3)$$

So associative.

Hence $\ker\varphi < G$.

Normal? $g \in G$, $g \ker\varphi = \ker\varphi\, g$?

let $k \in \ker\varphi$. Show $gk \in \ker\varphi \cdot g$

$$\varphi(gk) = \varphi(g)\cdot\varphi(k) = \varphi(g) = \varphi(k)\varphi(g) = \varphi(kg)$$

So $g \ker\varphi = \ker\varphi \cdot g$ as required

$\square$

We connect each hom to its kernel, a normal subgp.

<u>Theorem</u> (First isomorphism theorem)

Let $\varphi: G \to H$ be a surjective hom.

Consider $\psi: G/\ker\varphi \to H$ sending each coset $g \cdot \ker\varphi$ to $\varphi(g)$. Then ψ is an isomorphism.

Note that for this to be true we must have $\varphi(g_1) = \varphi(g_2)$ for any elements g_1 & g_2 of a single coset of $\ker\varphi$ (i.e. $g_1 \in g_2 \cdot \ker\varphi$).

HW: Prove that

<u>Proof</u>: (1st iso. thm.)

i) ψ injective: Let $g_1, g_2 \in G$

$\psi(g_1 \cdot \ker\varphi) = \psi(g_2 \ker\varphi)$ iff

$\varphi(g_1) = \varphi(g_2)$ iff

$g_1 \cdot \ker\varphi = g_2 \cdot \ker\varphi$ (HW) <u>injective</u>

ii) ψ surjective: φ surjective so $\forall\, h \in H\, \exists\, g \in G$ s.t. $\varphi(g) = h$. Then $\psi(g \cdot \ker\varphi) = \varphi(g) = h$ <u>surjective</u>

iii) ψ hom: $\psi(g_1 \cdot g_2 \ker\varphi) = \varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$

$= \psi(g_1 \ker\varphi) \cdot \psi(g_2 \ker\varphi)$ <u>homomorphism</u>

$\square$

<u>Remark</u>: One can understand homomorphisms $\varphi: G \to H$ w/o looking closely at the structure of H, just by looking at how $G \to G/\ker\varphi$ behaves. These are known as Noether's iso. thms after Emmy Noether.

Our next theorem says we gain no extra information by iterating this process w/ subgps

Theorem (Second isomorphism thm) Sometimes the third

If $N \lhd H \lhd G$ then $H/N \lhd G/N$ &

$$\frac{G/N}{H/N} \cong G/H$$

Proof: Let $\varphi : {}^{G/N}\!/_{H/N} \to G/H : (gN)(H/N) \mapsto gH \quad \forall g \in G$

$\ker \varphi = \{ g \in G \mid gH = e \} = H$

Finally we have the third isomorphism thm, which tells us what happens to subgps of G under the natural hom $G \to G/N$, $N \lhd G$.
First a definition.
Def^n. A product of two sets A & B is
$$AB = \{ab \mid a \in A, b \in B\}$$

Theorem (Third isomorphism thm)
Let $N \lhd G$, $H < G$, & $\varphi: G \to G/N$ be the natural homomorphism. Then

i) $HN < G$ & $N \lhd HN$
ii) $N \cap H \lhd H$

$\Big\}$ housekeeping

iii) $\varphi(H) = HN/N \cong H/(N \cap H)$

Essentially, restricting the natural hom. to a subgp yields no extra structural information.

Proof: i) + ii) are homework

iii) $\varphi(H) = \{hN \mid h \in H\} \cong HN/N \cong H/N \cap H$

$f : HN \to HN/N$ is the natural hom

Then $f|_H : H \to HN/N$ also a hom.

$\ker f = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$

So 1st. iso says

$$H/H \cap N \cong HN/N \qquad \square$$

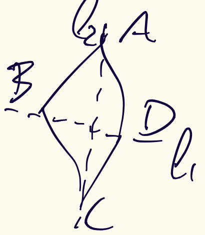
DIRECT
PRODUCT

Direct product

We have seen in the third isomorphism thm that $H < G$ & $N \triangleleft G$ implies $HN < G$. We present another important notion of product.

Defn: The <u>direct product</u> of two groups G & H is the group $G \times H$ of ordered pairs (g,h), $g \in G$ & $h \in H$. The binary operation sends $((g_1,h_1)\cdot(g_2,h_2))$ to (g_1g_2, h_1h_2) where the product g_1g_2 is in G & h_1h_2 is in H.

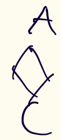
Example: The symmetries of a rhombus are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein 4 grp).

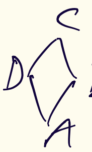


& R_1 = Rotation by π

Note $R_1^2 = F_1^2 = F_2^2 = e$

$R_1 \cdot F_1 = R_1$ 

$=$ 

$F_1 \cdot R_1 = F_1$ 

$=$ 

$R_1F_1 = F_1R_1 = F_2$

$R_1F_2 = F_2R_1 = F_1$

$F_1F_2 = F_2F_1 = R_1$

$e \longmapsto (0,0)$

$F_1 \longmapsto (1,0)$

$F_2 \longmapsto (0,1)$

$R_1 \longmapsto (1,1)$

HW show bijection & hom.

<u>Theorem</u> (Properties of direct product)

Let G & H be groups. Then

i) $G \times H$ is a group

ii) G, H abelian $\Rightarrow$ $G \times H$ abelian

iii) $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ iff $(m,n) = 1$